On the Fourier Constants of a Function.

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§ 1. Considerable progress has been made lately in the study of the properties of the constants in a Fourier series, using this term in the most general sense possible consistent with the extended definition of integration due to Lebesgue. Thus we now know that these coefficients necessarily under all circumstances have the unique limit zero,* as the integer denoting their place in the series increases indefinitely, and that the same is true if we substitute for that integer any other quantity which increases without limit. Further, we know that the series whose general term is b_n/n, where b_n is the typical coefficient of the sine terms, always converges, and we are able to write down its sum.† That the series whose general term is a_n, where a_n is the typical coefficient of the cosine terms, converges when the origin is an internal point of an interval throughout which the function has bounded variation, and that accordingly the series whose general term is a_n/n^q, $(0<q)$, converges, is an immediate consequence of known results. Should the function have its square summable,‡ we know§ that the series whose general term is $(a_n^2+b_n^2)$ converges, and we can write down its sum. We can also sum the series of the products of the Fourier coefficients of two such functions. From the property that $\Sigma(a_n^2+b_n^2)$ converges, we can deduce that the series $\Sigma\, a_n/n^q$ and $\Sigma\, b_n/n^q$, $(\frac{1}{2}<q)$, necessarily converge absolutely.

Again, making use of a theorem recently proved,‖ we may integrate the Fourier series of any summable function, after multiplying it term by term by any function of bounded variation, with a certainty that we shall obtain the same result as if the Fourier series converged to the function to which it corresponds, and such term-by-term integration were allowable.

Choosing for the function of bounded variation in question various simple

* B. Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," § 10, 'Ges. Werke,' 2nd edition, 1854, p. 254; H. Lebesgue, 'Leçons sur les séries trigonométriques," 1906, p. 61.

† H. Lebesgue, *loc. cit.*, § 53, p. 102.

‡ A function is, in accordance with the usage now accepted, said to be summable when it possesses a Lebesgue integral proper or improper. This has superseded the nomenclature in Hobson's 'Treatise on the Theory of Functions of a Real Variable.'

§ P. Faton, 'Séries trigonométriques et séries de Taylor," 'Acta. Math.,' 1905, vol. 30, pp. 335—400.

‖ W. H. Young, "On the Integration of Fourier Series," 1910, presented to the L.M.S.

functions we shall obtain the sums of a number of series involving the Fourier coefficients of any summable function.

By using the other theorems in the paper last cited we obtain other results of a less general character.

In the present note I propose to employ the theorems of that paper to prove the following properties of the coefficients:—

(1) Whatever be the nature of the summable function $f(x)$, provided only that in the neighbourhood of the origin it is of bounded variation, then not only as is obvious the series of the a_n's, but also the series of the b_n's, is convergent when the individual terms of the series are divided by any the same positive power, however small, of the integer denoting the place of the coefficient in the Fourier series; further, the sums of the series are expressible as simple integrals.*

(2) Whatever be the nature of the summable function $f(x)$, provided only that in the neighbourhood of the origin its square is summable, the same property holds good, so long as the index of the power of the dividing integer is greater than $\frac{1}{2}$; the sums of the series are of the same form as in (1).

(3) If in the neighbourhood of the origin we can only assert that the function is bounded, the statement remains true if we interpret the terms "convergence" and "sum" both in the Cesaro sense.

(4) If in the neighbourhood of the origin one of the three preceding conditions holds, while in the rest of the interval $(-\pi, \pi)$ $f(x)$ possesses a Harnack-Lebesgue integral, the corresponding statement is true, if we interpret the terms "convergence" and "sum" in the Cesaro sense. The Fourier series is then a generalised one.

Closely associated with the first of these four statements is the following which is also proved below:—

(1 *bis*) Whatever be the nature of the summable function $f(x)$, provided that in some interval containing the origin the function $[f(x)-f(-x)]/x$ is summable, the series whose general term is $n^{-q}b_n$, $(0 \leq q < 1)$, is convergent; if, on the other hand, for some value of the constant C, the function $[f(x)+f(-x)-C]/x$ is summable in some interval containing the origin, the series whose general term is $n^{-q}a_n$, $(0 < q < 1)$, is convergent; in both cases it is to be supposed that in the rest of the interval $(-\pi, \pi)$ the function $f(x)$ is summable. Moreover, the formulæ for the sums have the same form as in (1).

The first part of the statement (1 *bis*) just made contains a result which seems to have a special interest of its own. As is well known, the series

* See below § 3, formulæ I and II; § 5, formula III; § 6, formula IV; §§ 7 and 8.

whose general term is a_n converges, and has $\frac{1}{2}[f(+0)-f(-0)]$ for its sum when $f(x)$ is a function of bounded variation. The result in question is what may accordingly be called the companion result, namely,* that the series whose general term is b_n converges, and has $\frac{1}{\pi}\int_0^\infty \frac{f(x)-f(-x)}{x}\,dx$ for its sum, provided only that this integral exists. Here it must be recollected that $f(x)$ has been made periodic in the usual way, so that the integral in question certainly exists between any limits of integration which do not contain the origin between them; thus the only limitation of the nature of the function $f(x)$ relates to the neighbourhood of the origin.

The results have all been stated with reference to the coefficients and not to the Fourier series itself. It should, however, be hardly necessary to remark that, by transforming the origin, we obtain corresponding results relating to the Fourier series itself, and to its very important, up to the present almost entirely neglected, companion series, got from it by interchanging the coefficients of the sine and cosine terms, and changing the sign of the latter. In particular we have in this way the sum of the companion series, when a certain function is summable; we can also find the function of which this series is the Fourier series in an important class of cases, in which we know it to be a Fourier series. I reserve for subsequent publication the further development of this remark.

In conclusion I would add that the formulæ obtained may be generalised in a variety of ways by means of the theorems here utilised. The present paper, however, contains the complete solution, within the limits proposed, of the problem with which we started, thus illustrating the usefulness of the theorems in the paper so frequently quoted.

§ 2. Denote by $f_1(x)$ the function which is zero at the extremities of the interval $(-\pi, \pi)$, and which, inside this interval, is equal to the expression

$$\tfrac{1}{2}[f(x)+f(-x)]-\frac{1}{4\pi}\int_{-\pi}^{\pi}[f(x)+f(-x)]\,dx,$$

while, outside this interval, such values are to be assigned to $f_1(x)$ as to make it periodic with period 2π.

Denote by $f_2(x)$ the function got from $f_1(x)$ by changing $f(-x)$ into $-f(-x)$. Then evidently $f_1(x)$ and $f_2(x)$ are summable, have their squares summable, are bounded, or of bounded variation, in a neighbourhood containing the origin, if $f(x)$ possesses these respective properties.

Further, since $\int_{-\pi}^{\pi} f_1(x)\,dx$ and $\int_{-\pi}^{\pi} f_2(x)\,dx$ are both zero, it is clear that the

* § 9, formula V.

indefinite integrals of $f_1(x)$ and $f_2(x)$ will all be periodic, and accordingly oscillate finitely as the upper limit of integration approaches infinity.

Again, as we have just seen, the Fourier series corresponding to $f_1(x)$ and $f_2(x)$ will be deficient of constant term. Moreover,

$$\int_{-\pi}^{\pi} f_1(x) \cos nx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} [f(x)+f(-x)-c_1] \cos nx\, dx = \int_{-\pi}^{\pi} f(x) \cos nx\, dx = a_n,$$

where c_1 is a constant.

Also $$\int_{-\pi}^{\pi} f_1(x) \sin nx\, dx = \tfrac{1}{2}\int_{-\pi}^{\pi} [f(x)+f(-x)-c_1] \sin nx\, dx = 0.$$

Similarly, $$\int_{-\pi}^{\pi} f_2(x) \cos nx\, dx = 0, \qquad \int_{-\pi}^{\pi} f_2(x) \sin nx\, dx = b_n.$$

§ 3. We first consider the case where $f(x)$ has bounded variation in the neighbourhood of the origin, and the index q of the power to which the integer n is raised lies between zero and unity, the extreme values not included.

Put $g(x) = x^{q-1}$, where $0<q<1$, and consider the integral $\int_0^{\infty} g(x) f_1(x)\, dx$. Except in the neighbourhood of the origin, $g(x)$ is a function of bounded variation in the whole infinite interval, and has, as x increases indefinitely, the unique limit zero. Moreover, $f_1(x)$ is such that $\int_k^x f_1(x)\, dx$ is a periodic function of x which oscillates finitely in the whole infinite interval (k, ∞). Hence, by the extension to infinity of the theorem cited in § 1, $\int_k^{\infty} g(x) f_1(x)\, dx$ is the sum of the series of integrals got by integrating, between the limits k and infinity, the successive terms of the Fourier series of $f_1(x)$, after having previously multiplied them by $g(x)$.

Next consider $\int_0^k g(x) f_1(x)\, dx$. In the interval of integration $g(x)$ is summable, while $f_1(x)$ has bounded variation, provided our choice of k has been a suitable one, which we may suppose to be the case. To this integral we may apply the theorem already cited in its simple form, so that we see that it is equal to the sum of the series of integrals got by integrating between the limits 0 and k the successive terms of the Fourier series of $f_1(x)$, after having previously multiplied them by $g(x)$.

Adding the two results so obtained, and bearing in mind that the Fourier series of $f_1(x)$ has no sine terms, we have

$$\int_0^{\infty} x^{q-1} f_1(x)\, dx = \sum_{n=1}^{\infty} a_n \int_0^{\infty} x^{q-1} \cos nx\, dx.$$

Similarly $$\int_0^\infty x^{q-1} f_2(x)\,dx = \sum_{n=1}^\infty b_n \int_0^\infty x^{q-1} \sin nx\,dx.$$

Hence, since we know that, when $0 < q < 1$,

$$\int_0^\infty x^{q-1} \cos x\,dx = \Gamma(q) \cos \tfrac{1}{2} q\pi \quad \text{and} \quad \int_0^\infty x^{q-1} \sin x\,dx = \Gamma(q) \sin \tfrac{1}{2} q\pi,$$

we get

$$\int_0^\infty x^{q-1} f_1(x)\,dx = \sum_{n=1}^\infty a_n \int_0^\infty \left(\frac{x}{n}\right)^{q-1} \cos x\, d\left(\frac{x}{n}\right) = \sum_{n=1}^\infty \frac{a_n}{n^q} \int_0^\infty x^{q-1} \cos x\,dx$$

$$= \cos \tfrac{1}{2} q\pi \Gamma(q) \sum_{n=1}^\infty n^{-q} a_n. \tag{I}$$

Similarly $$\int_0^\infty x^{q-1} f_2(x)\,dx = \sin \tfrac{1}{2} q\pi\, \Gamma(q) \sum_{n=1}^\infty n^{-q} b_n. \tag{II}$$

§ 4. Before going on to obtain the corresponding formulæ when q is not internal to the interval $(0, 1)$, we remark that the results above obtained are obviously true, *mutatis mutandis*, when near the origin $f(x)$ is respectively bounded, or has its square summable. In the former case the above argument is unaffected, provided only we interpret the words "sum" and "convergence" of a series in the Cesaro manner. In fact, the reasoning for the interval (k, ∞) applies, as before, as all that was required of $f(x)$ in that interval was that it should be summable in every finite portion of it. As regards the interval $(0, k)$, the theorem used is still applicable, as is shown in the paper quoted, with the proviso in question.

If, on the other hand, all we know of $f(x)$, and therefore of $f_1(x)$ and $f_2(x)$, in the neighbourhood of the origin, is that its square is summable, we need the summability of the square of $g(x)$ in the interval $(0, k)$. This requires q to be greater than $\frac{1}{2}$. With this understanding, therefore, our results are still true.

If, finally, we only know of $f(x)$ near $x = \pi$ or $-\pi$ that it possesses a Harnack-Lebesgue integral, the corresponding portion of the integrals will in general only converge in the Cesaro way.* Hence in this case, even if $f(x)$ is of bounded variation near the origin, the final series can only be summed in general in the Cesaro way.

§ 5. Let us now make the hypothesis that q is unity. By the theory of the integration of Fourier series, we have, if $F_1(x)$, $F_2(x)$, denote any definite integrals of $f_1(x), f_2(x)$,

$$F_1(x) = \text{const.} + \sum_{n=1}^\infty n^{-1} a_n \sin nx,$$

$$F_2(x) = \text{const.} - \sum_{n=1}^\infty n^{-1} b_n \cos nx.$$

* "On the Integration of Fourier Series," § 7.

The first equation gives us no information, the second equation tells us that

$$\sum_{n-1}^{\infty} n^{-1} b_n = \text{const.} - \mathrm{F}_2(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{F}_2(x)\,dx - \mathrm{F}_2(0). \tag{III}$$

Here the summation must be supposed performed in the Cesaro manner, when the Fourier series is a generalised one.*

This result is due to Lebesgue,† in the case when the Fourier series is an ordinary one.

§ 6. To obtain the sum of the series whose general term is $n^{-1}a_n$, we must adopt a different method. We require to use the fact that, in the neighbourhood of the origin, $f(x)$ has, under all the circumstances supposed, its square summable. On the other hand the function

$$\tfrac{1}{2}\log \tfrac{1}{4}\operatorname{cosec}^2 \tfrac{1}{2}x$$

has, except in the neighbourhood of the origin, bounded variation throughout the interval $(-\pi, \pi)$. In the neighbourhood of the origin it has its square summable—in fact, $\log x - \log \sin x < \log (1+\tfrac{1}{6}x)^2 < x^2/6 < x^2$.

Therefore, $$(\log \operatorname{cosec} x)^2 < x^4 - 2x^2 \log x + (\log x)^2.$$

Hence, also, the square of the function in question, $\tfrac{1}{2}\log\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}x$, is less than a summable function.

Thus the interval $(-\pi, \pi)$ can be broken up into two parts, in one of which one of the two functions $f_1(x)$ and $\tfrac{1}{2}\log\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}x$ is of bounded variation and the other is summable, while in the remaining part both the functions have their squares summable. Applying, therefore, the theorems already referred to, to these separate portions, and adding the results, we get

$$\int_{-\pi}^{\pi} f_1(x)\,\tfrac{1}{2}\log\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}x\,dx = \sum_{n=1}^{\infty} n^{-1}a_n, \tag{IV}$$

since the generic Fourier constant of the function $\tfrac{1}{2}\log\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}x$ is n^{-1}.‡

Here we have assumed $f(x)$ to be summable. If, in any portion of the interval $(-\pi, \pi)$, it have only a Harnack-Lebesgue integral, the argument still applies, provided we sum the series in the Cesaro way.

§ 7. Since the function corresponding to an integrated Fourier series is an integral, it is, of course, a continuous function, and belongs also, *a fortiori*, to the class of functions whose square is summable. Accordingly,

* W. H. Young, "On the Conditions that a Trigonometrical Series should have the Fourier Form," 1910. Supplementary Note. Presented to the London Mathematical Society.

† *Loc. cit.*

‡ D. Bernoulli, 'Petrop. N. Comm.,' 1772. See Hobson's 'Theory of Functions of a Real Variable,' p. 639.

the method of § 6 enables us to sum the series, whose general terms are respectively

$$n^{-2}b_n,\ n^{-3}a_n\ldots,$$

where a_n and b_n are the Fourier constants of any summable function whatever. The formulæ, of course, closely resemble that obtained at the conclusion of that article.

On the other hand, the series whose general terms are

$$n^{-2}a_n,\ n^{-3}b_n\ldots,$$

where, again, a_n and b_n are the Fourier constants of any summable function whatever, are obtained by mere repetition of the process employed in §5, their sums having a form which closely resembles that of the series summed in that article.

§ 8. The case in which q is greater than unity, and not an integer, alone remains to be considered. The method to be adopted is obvious. We have only to apply to the integrated series the reasoning of § 3, with this simplification, that the function to which the Fourier series corresponds, being now an integral of $f_1(x)$, or of $f_2(x)$, as the case may be, is certainly of bounded variation in the case of an ordinary Fourier series, and at least continuous in the case of a generalised Fourier series in every finite interval, so that no special hypothesis is needed. The formulæ are easily obtained.

§ 9. To write down (by request) the formulæ referred to in §§ 7 and 8, it is convenient to use the following notation:—

$$f_1(x) = \tfrac{1}{2}\{f(x)+f(-x)\}-\frac{1}{2\pi}\int_0^\pi \{f(x)+f(-x)\}\,dx,$$

$$f_2(x) = \tfrac{1}{2}\{f(x)-f(-x)\},$$

$$F_1(x) = \int_0^x f_1(x)\,dx, \qquad F_2(x) = \int_0^x f_2(x)\,dx,$$

$$G_1(x) = \int_0^x F_1(x)\,dx, \qquad G_2(x) = \int_0^x \left\{F_2(x)-\frac{1}{\pi}\int_0^\pi F_2(x)\,dx\right\}dx,$$

$$H_1(x) = \int_0^x \left\{G_1(x)-\frac{1}{\pi}\int_0^\pi G_1(x)\,dx\right\}dx, \qquad H_2(x) = \int_0^x G_2(x)\,dx,$$

$$K_1(x) = \int_0^x H_1(x)\,dx, \qquad K_2(x) = \int_0^x H_2(x)\,dx,$$

and so on.

We then have the following formulæ, which are those given in § 5 and the analogous formulæ referred to at the end of §7:—

$$f_1(x) \sim \sum_{n=1}^{\infty} a_n \cos nx, \qquad f_2(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \qquad F_1(x) = \sum_{n=1}^{\infty} n^{-1}a_n \sin nx,$$

$$\mathrm{F}_2(x)-\frac{1}{\pi}\int_0^\pi \mathrm{F}_2(x)\,dx = -\sum_{n=1}^{\infty} n^{-1}b_n\cos nx, \quad \text{thus} \quad \sum_{n=1}^{\infty} n^{-1}b_n = \frac{1}{\pi}\int_0^\pi \mathrm{F}_2(x)\,dx,$$

$$\mathrm{G}_1(x)-\frac{1}{\pi}\int_0^\pi \mathrm{G}_1(x)\,dx = -\sum_{n=1}^{\infty} n^{-2}a_n\cos nx, \quad \text{thus} \quad \sum_{n=1}^{\infty} n^{-2}a_n = \frac{1}{\pi}\int_0^\pi \mathrm{G}_1(x)\,dx,$$

$$\mathrm{G}_2(x) = -\sum_{n=1}^{\infty} n^{-2}b_n\sin nx, \qquad \mathrm{H}_1(x) = -\sum_{n=1}^{\infty} n^{-3}a_n\sin nx,$$

$$\mathrm{H}_2(x)-\frac{1}{\pi}\int_0^\pi \mathrm{H}_2(x)\,dx = \sum_{n=1}^{\infty} n^{-3}b_n\cos nx, \quad \text{thus} \quad \sum_{n=1}^{\infty} n^{-3}b_n = -\frac{1}{\pi}\int_0^\pi \mathrm{H}_2(x)\,dx,$$

$$\mathrm{K}_1(x)-\frac{1}{\pi}\int_0^\pi \mathrm{K}_1(x)\,dx = \sum_{n=1}^{\infty} n^{-4}a_n\cos nx, \quad \text{thus} \quad \sum_{n=1}^{\infty} n^{-4}a_n = -\frac{1}{\pi}\int_0^\pi \mathrm{K}_1(x)\,dx,$$

$$\mathrm{K}_2(x) = \sum_{n=1}^{\infty} n^{-4}b_n\sin nx,$$

and so on.

The formula obtained in § 6,

$$\int_{-\pi}^{\pi} f_1(x)\,\tfrac{1}{2}\log\tfrac{1}{4}\operatorname{cosec}^2\tfrac{1}{2}x\,dx = \sum_{n=1}^{\infty} n^{-1}a_n,$$

may evidently be written

$$2\int_0^\pi f_1(x)\log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\,dx = \sum_{n=1}^{\infty} n^{-1}a_n.$$

Similarly, without any restriction on $f(x)$, with the notation just explained, bearing in mind that $\int_0^\pi \log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\,dx = 0$, we have the analogous formulæ referred to at the beginning of § 7:—

$$2\int_0^\pi \mathrm{F}_2(x)\log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\,dx = -\sum_{n=1}^{\infty} n^{-2}b_n,$$

$$2\int_0^\pi \mathrm{G}_1(x)\log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\,dx = -\sum_{n=1}^{\infty} n^{-3}a_n,$$

$$2\int_0^\pi \mathrm{H}_2(x)\log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\,dx = \sum_{n=1}^{\infty} n^{-4}b_n,$$

$$2\int_0^\pi \mathrm{K}_1(x)\log\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}x\,dx = \sum_{n=1}^{\infty} n^{-5}a_n,$$

and so on.

The formulæ referred to in § 8, in which $0 < q < 1$, are then as follows:—

$$\int_0^\infty x^{q-1}F_1(x)\,dx = \sin\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{\pi=1}^\infty n^{-q-1}a_n,$$

$$\int_0^\infty x^{q-1}\left[F_2(x)-\frac{1}{\pi}\int_0^\pi F_2(x)\,dx\right]dx = -\cos\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^\infty n^{-q-1}b_n,$$

$$\int_0^\infty x^{q-1}\left[G_1(x)-\frac{1}{\pi}\int_0^\pi G_1(x)\,dx\right]dx = -\cos\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^\infty n^{-q-2}a_n,$$

$$\int_0^\infty x^{q-1}\,G_2(x)\,dx = -\sin\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{=1}^\infty n^{-q-2}b_n,$$

$$\int_0^\infty x^{q-1}\,H_1(x)\,dx = -\sin\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^\infty n^{-q-3}a_n,$$

$$\int_0^\infty x^{q-1}\left[H_2(x)-\frac{1}{\pi}\int_0^\pi H_2(x)\,dx\right]dx = \cos\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^\infty n^{-q-3}b_n,$$

$$\int_0^\infty x^{q-1}\left[K_1(x)-\frac{1}{\pi}\int_0^\pi K_1(x)\,dx\right]dx = \cos\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^\infty n^{-q-4}a_n,$$

$$\int_0^\infty x^{q-1}\,K_2(x)\,dx = \sin\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^\infty n^{-q-4}b_n,$$

and so on.

§ 10. We now assume that $\{f(x)-f(-x)\}/x$ is summable in an interval containing the origin, and proceed to prove the results (1 *bis*) of the introduction.

Since $[f(x)-f(-x)]/x$ is, by hypothesis, a summable function, so is $[f(x)-f(-x)]/\sin x$. Also $\sin x/x$ is a function of bounded variation in any finite interval $(0, k)$. Hence, using the theorem so often quoted, and denoting by πA_n the integral $\int_{-\pi}^{\pi}[f(x)-f(-x)]\operatorname{cosec} x\cos nx\,dx$, we have

$$\int_0^k \frac{f(x)-f(-x)}{\sin x}\cdot\frac{\sin x}{x}\,dx = \tfrac{1}{2}\int_0^k A_0\frac{\sin x}{x}\,dx + \sum_{n=1}^\infty \int_0^k A_n\frac{\cos nx\sin x}{x}\,dx; \qquad (1)$$

so that, in particular, the series on the right-hand side converges.

Now the summation on the right-hand side of the preceding equation is the unique limit when m increases indefinitely of the following:—

$$\tfrac{1}{2}\sum_{n=1}^m \int_0^k A_n\frac{\sin(n+1)x-\sin(n-1)x}{x}\,dx = -\tfrac{1}{2}A_2\int_0^k\frac{\sin x}{x}\,dx$$
$$+\tfrac{1}{2}A_{m-1}\int_0^k\frac{\sin mx}{x}\,dx+\tfrac{1}{2}A_m\int_0^k\frac{\sin(m+1)x}{x}\,dx$$
$$+\tfrac{1}{2}\sum_{n=1}^{m-2}(A_n-A_{n+2})\int_0^k\frac{\sin(n+1)x}{x}\,dx. \qquad (2)$$

But $$A_n - A_{n+2} = \frac{1}{\pi}\int_{-\pi}^{\pi} 2[f(x)-f(-x)]\sin(n+1)x\,dx = 4b_{n+1}.$$

Hence (1) becomes

$$\tfrac{1}{2}\int_0^k \frac{f(x)-f(-x)}{x}\,dx = \sum_{n=1}^{\infty}\int_0^k b_n \frac{\sin nx}{x}\,dx, \tag{3}$$

for in (2) the second and third terms on the right vanish when we proceed to the limit, since A_{m-1} and A_m, being the Fourier coefficients of a summable function, have zero as unique limit when m increases indefinitely, while the integrals by which they are multiplied are always numerically less than π. But, by the usual argument,

$$\tfrac{1}{2}\int_k^{\infty} \frac{f(x)-f(-x)}{x}\,dx = \sum_{n=1}^{\infty}\int_k^{\infty} b_n \frac{\sin nx}{x}\,dx. \tag{4}$$

Hence, adding (3) and (4), and using the fact that $\int_0^{\infty}\frac{\sin nx}{x}\,dx = \frac{\pi}{2}$, we have

$$\tfrac{1}{2}\int_0^{\infty} \frac{f(x)-f(-x)}{x}\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} b_n,$$

which may, if $f(x)/x$ is summable, be written in the form

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(x)}{x}\,dx = \sum_{n=1}^{\infty} b_n. \tag{V}$$

§ 11. A slight modification of the argument used in the preceding article enables us to show that our formulæ (1) and (2) hold on the new hypothesis of the introduction, q lying, as before, in the open interval (0, 1). In formula (2) we require $f_2(x)/x$ to be summable in the neighbourhood of the origin, while in formula (I) the condition is that $(f_1(x)-C)/x$ should for some value of the constant C be summable in the neighbourhood of the origin.

To prove this we work with the factor $x^{q-1}\sin x$, instead of $x^{-1}\sin x$, and arrive, by the same reasoning as in § 10, at the equation corresponding to (3), viz.:—

$$\tfrac{1}{2}\int_0^k x^{q-1}[f(x)-f(-x)]\,dx = \sum_{n=1}^{\infty}\int_0^k b_n x^{q-1}\sin nx\,dx.$$

Adding to this the equation corresponding to (4), we get, as before,

$$\tfrac{1}{2}\int_0^{\infty} x^{q-1}[f(x)-f(-x)]\,dx = \sum_{n=1}^{\infty} b_n\int_0^{\infty} x^{q-1}\sin nx\,dx = \sum_{n=1}^{\infty} b_n n^{-q}\int_0^{\infty} x^{q-1}\sin x\,dx.$$

$$= \cos\tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^{\infty} n^{-q} b_n. \tag{II}$$

Again in like manner,

$$\int_0^1 \frac{f_1(x)-C}{\sin x}\,.\,x^{q-1}\sin x\,dx = \sum_{n=1}^{\infty} B_n\int_0^1 x^{q-1}\sin x\sin nx\,dx, \tag{1}$$

where $$\mathrm{B}_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (f_1(x)-\mathrm{C}) \operatorname{cosec} x \sin nx \, dx,$$

and therefore

$$\mathrm{B}_{n+1}-\mathrm{B}_{n-1} = 2a_n, \qquad \mathrm{B}_2 = 2a_1, \qquad \mathrm{B}_1 = -2\mathrm{C}. \tag{2}$$

Hence the summation on the right-hand side of the equation (1), just written down, is the unique limit when m increases indefinitely of one half of the following expression :—

$$2\sum_{n=1}^{n=m} \mathrm{B}_n \int_0^1 x^{q-1}[\cos(n-1)x-\cos(n+1)x]\,dx = \mathrm{B}_1\int_0^1 x^{q-1}\,dx+\mathrm{B}_2\int_0^1 x^{q-1}\cos x\,dx$$
$$-\mathrm{B}_{m-1}\int_0^1 x^{q-1}\cos mx\,dx-\mathrm{B}_m\int_0^1 x^{q-1}\cos(m+1)x\,dx$$
$$+\sum_{n=1}^{n=m-1} x^{q-1}(\mathrm{B}_{n+1}-\mathrm{B}_{n-1})\cos nx\,dx. \tag{3}$$

Hence, using (2), we get from (1) the following equation :—

$$\int_0^1 x^{q-1}f_1(x)\,dx = \sum_{n=1}^{\infty} a_n \int_0^1 x^{q-1}\cos nx\,dx,$$

since the second and third terms on the right-hand side of (3) have, as m increases indefinitely, the unique limit zero, B_m and B_{m-1} being the Fourier coefficients of a summable function.

But, by the usual argument,

$$\int_1^{\infty} x^{q-1}f_1(x)\,dx = \sum_{n=1}^{\infty} a_n \int_1^{\infty} x^{q-1}\cos nx\,dx.$$

Adding the last two equations, we get, as in § 3, the required result :—

$$\int_0^{\infty} x^{q-1}f_1(x)\,dx = \cos \tfrac{1}{2}q\pi\,\Gamma(q)\sum_{n=1}^{\infty} n^{-q}a_n. \tag{II}$$